Quantum Fall Festival @ Yonsei University

Basics of Quantum Informatics

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CONTENTS

01. Hamiltonian & Unitarty

What is hamiltonian?

What is Unitary?

Hamiltonian

Observable

Unitary

Hermitian

01. Hamiltonian & Unitarty

1) Hamiltonian

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t).$$

$$= \hat{H}$$

- Def

해밀토니언(Hamiltonian, $\hat{H}$ 또는 H로 표기)은 양자 상태의 시간 변화를 생성하는 에르미트 연산자이다.
이는 고전 해밀턴 역학에서 해밀토니언을 양자화하여 얻을 수 있고, 고전적인 에너지를 나타낸다.

- Hermitian (self-adjoint)

$$\hat{H} = \hat{H}^\dagger$$

$$\hat{H}|\Psi\rangle = E|\Psi\rangle$$

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01. Hamiltonian & Unitarty

2) Observable

$$\hat{H} = \hat{H}^\dagger \qquad \hat{H}|\Psi\rangle = E|\Psi\rangle$$

- Def

In physics, an **observable** is a physical property or physical quantity that can be measured. In classical mechanics, an observable is a real-valued "function" on the set of all possible system states, e.g., position and momentum. In quantum mechanics, an observable is an operator, or gauge, where the property of the quantum state can be determined by some sequence of operations. For example, these operations might involve submitting the system to various electromagnetic fields and eventually reading a value.

01. Hamiltonian & Unitarty

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2) Observable

$$\hat{H} = \hat{H}^\dagger \qquad \hat{H}|\Psi\rangle = E|\Psi\rangle$$

- cf)

Problem 2.1 Prove the following three theorems:

(a) For normalizable solutions, the separation constant E must be *real*. *Hint:* Write E (in Equation 2.7) as $E_0 + i\Gamma$ (with E_0 and Γ real), and show that if Equation 1.20 is to hold for all t, Γ must be zero.

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar},$$

$$\boxed{\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\, dx = 1.}$$

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01. Hamiltonian & Unitarty

3) Hermitian

$$\hat{H}|\Psi\rangle = E|\Psi\rangle$$

- Def

$$\hat{H} = \hat{H}^\dagger$$

- Real eigenvalue

$$\begin{aligned} E &= \langle\Phi|\hat{H}|\Phi\rangle = (\langle\Phi|\hat{H})|\Phi\rangle \\ &= (\hat{H}|\Phi\rangle)^\dagger|\Phi\rangle \\ &= \langle\Phi|\bar{E}|\Phi\rangle = \bar{E} \end{aligned}$$

01. Hamiltonian & Unitarty

4) Unitary

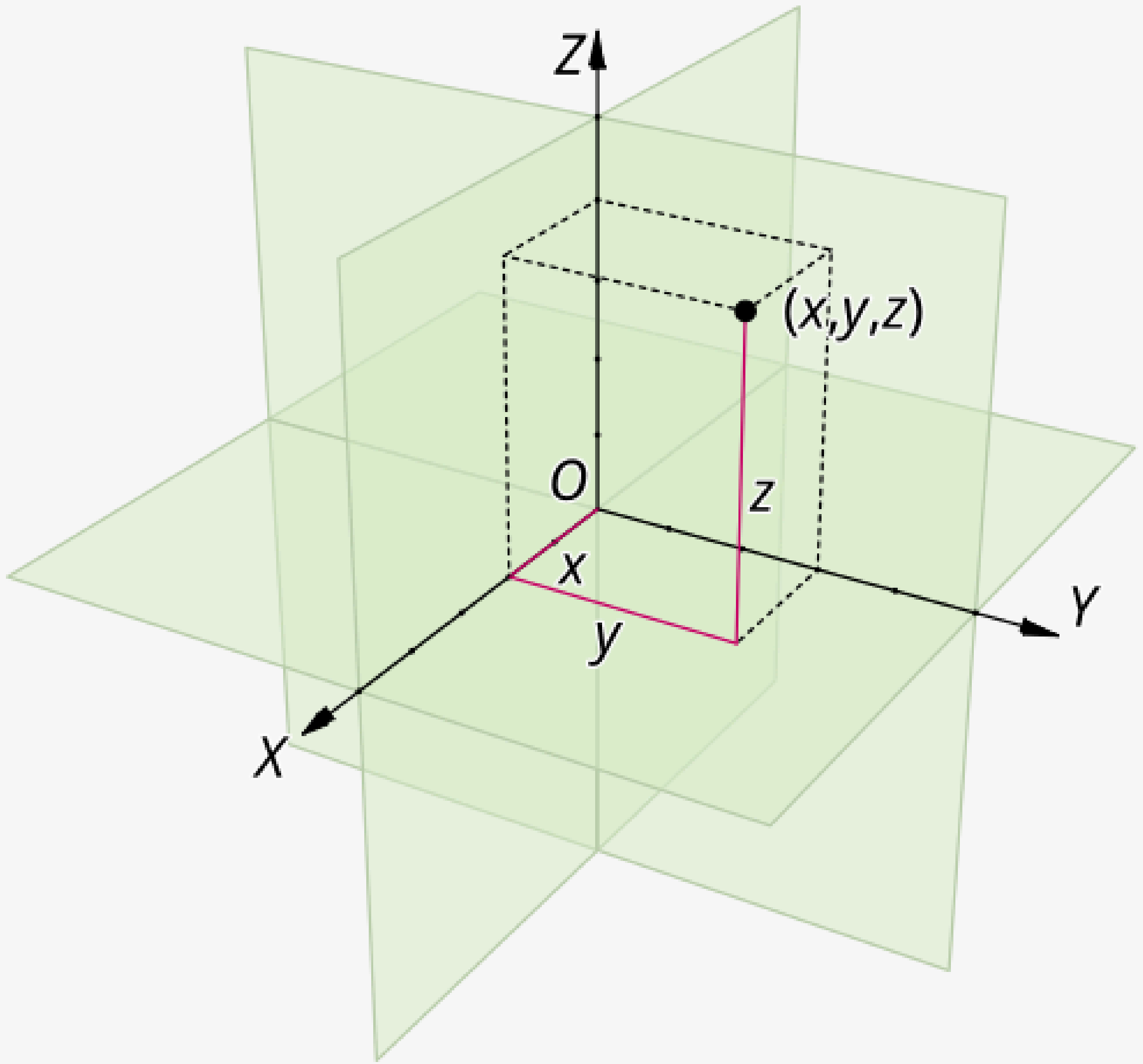
- Def

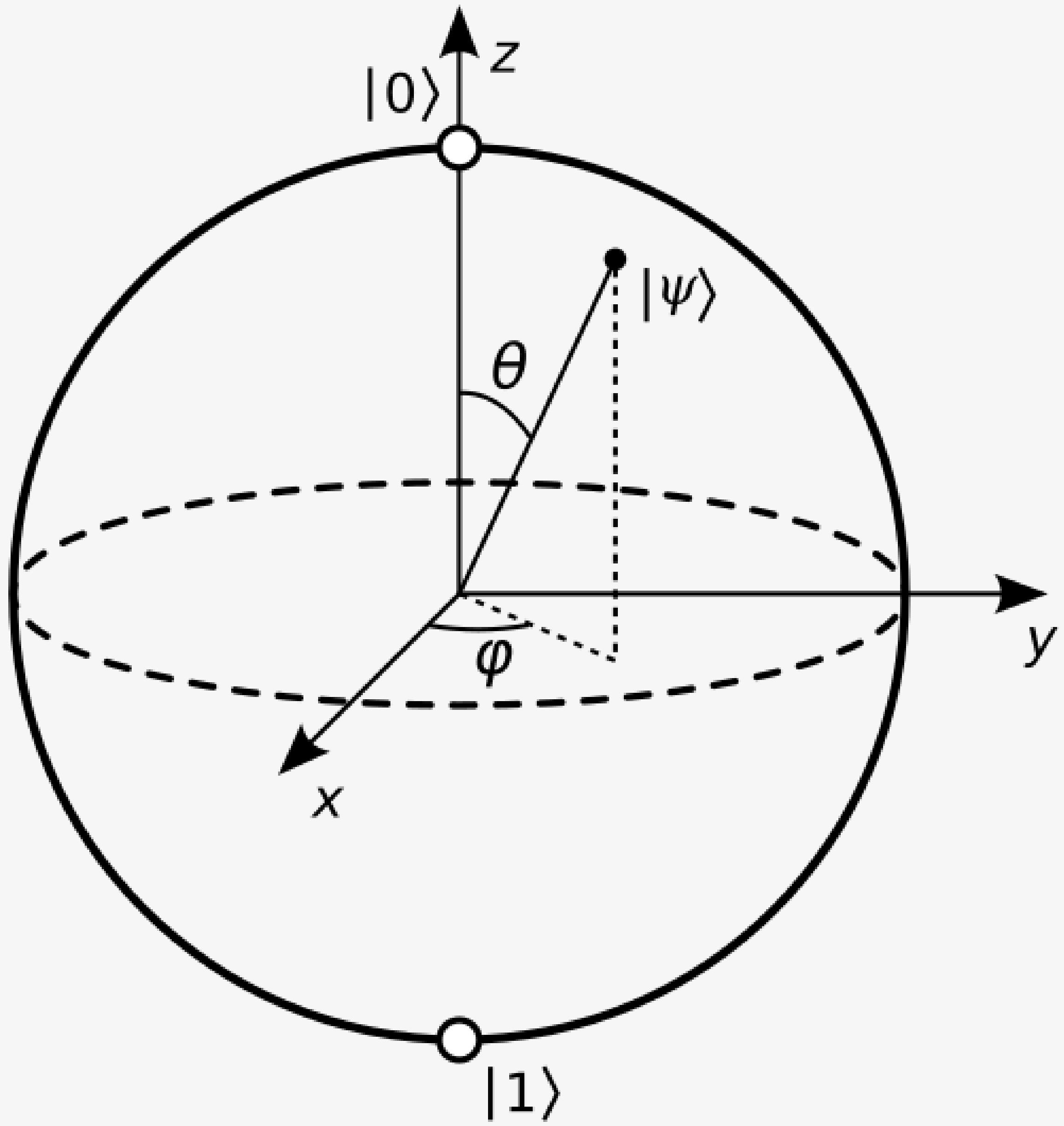
$$\hat{U}\hat{U}^\dagger = I \qquad \hat{U}^\dagger = \hat{U}^{-1}$$

Problem 6.2 Show that, for a Hermitian operator $\hat{Q}$, the operator $\hat{U} = \exp\left[i\,\hat{Q}\right]$ is unitary.
Hint: First you need to prove that the adjoint is given by $\hat{U}^\dagger = \exp\left[-i\,\hat{Q}\right]$; then prove that $\hat{U}^\dagger\,\hat{U} = 1$. Problem 3.5 may help.

02. Bloch sphere

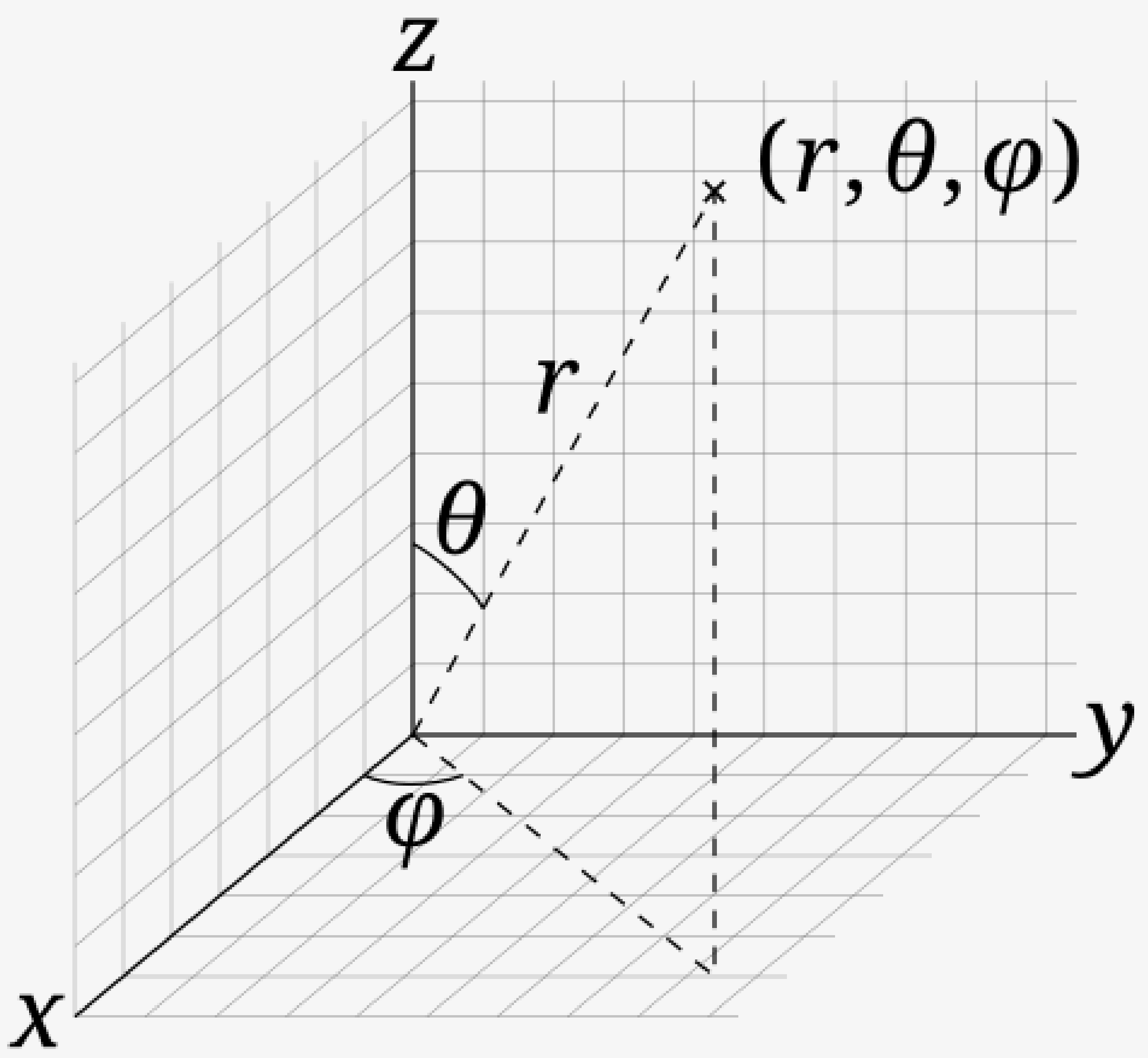
1) Cartesian Coord. vs. Bloch sphere

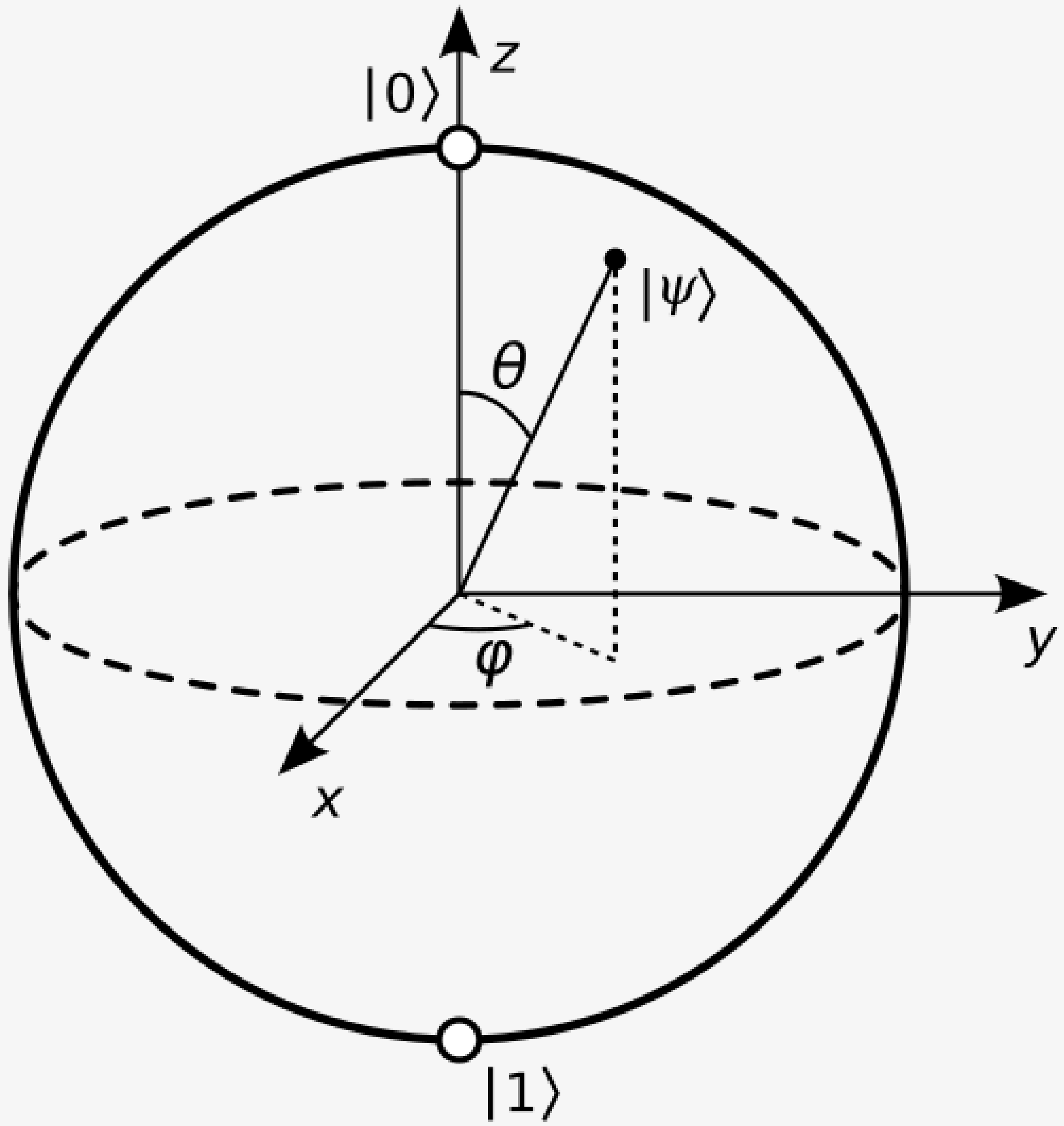




02. Bloch sphere

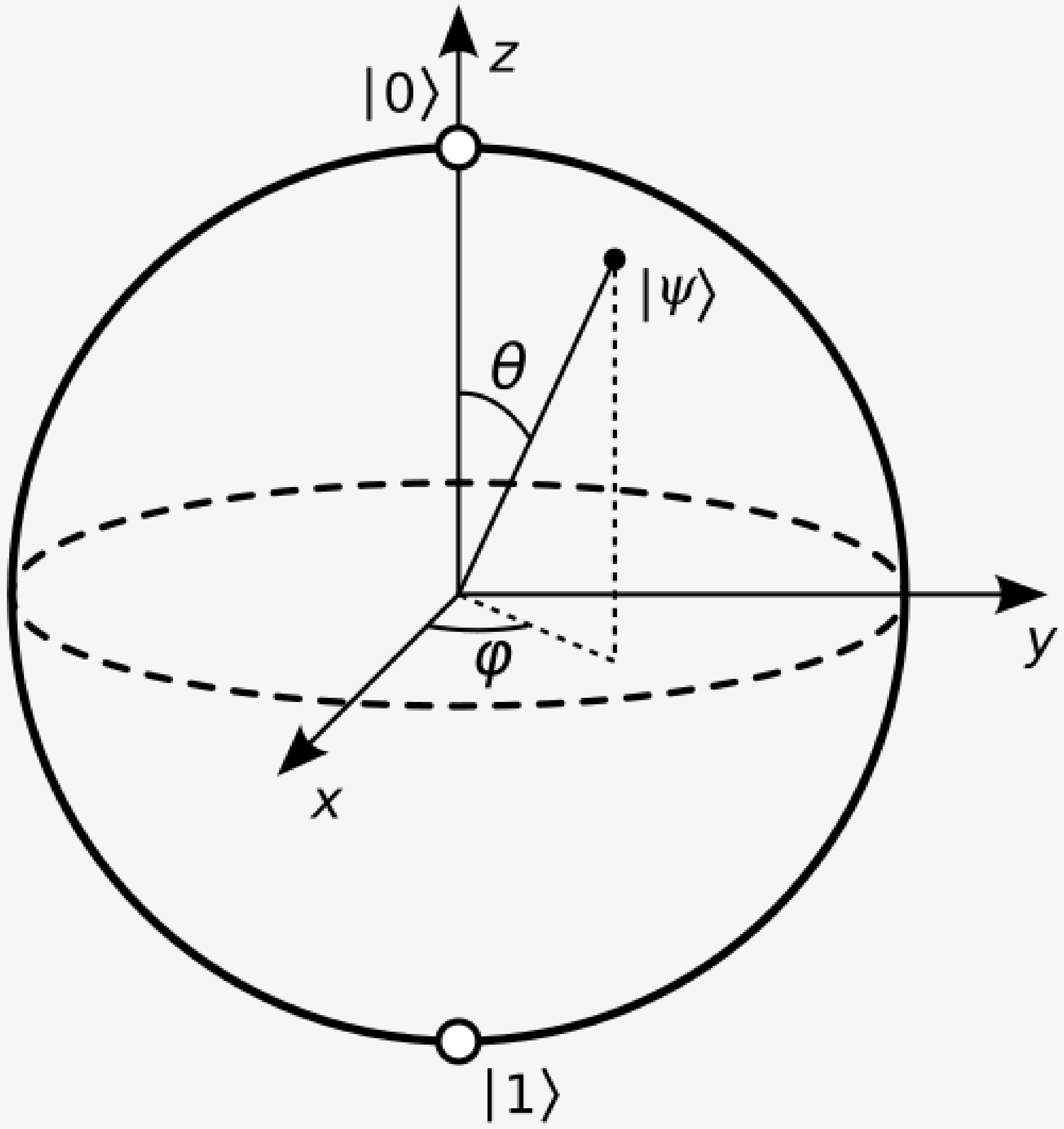
2) Spherical Coord. vs. Bloch sphere





02. Bloch sphere

3) Bloch sphere



z-axis

$$|0\rangle = |\uparrow\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

$$|1\rangle = |\downarrow\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

x-axis

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

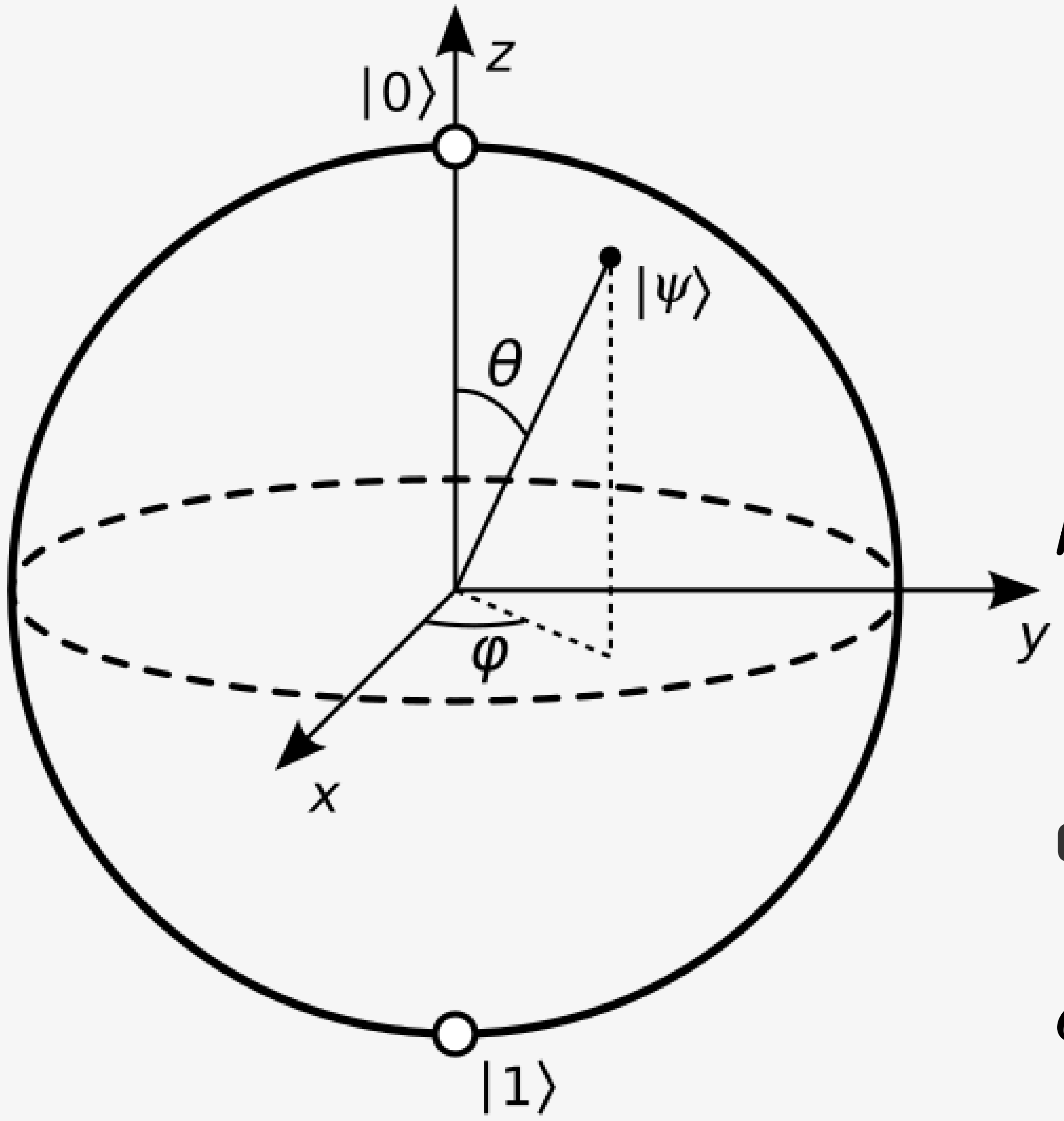
y-axis

$$|+i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix} = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

$$|-i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix} = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$

02. Bloch sphere

3) Bloch sphere (cont.)



$$\langle\psi|\psi\rangle = 1$$

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$ (surface-pure)

$$\rho = \frac{1}{2}(I + \vec{a}\cdot\vec{\sigma}) = \begin{bmatrix} 1+a_z & a_x - ia_y \\ a_x + ia_y & 1-a_z \end{bmatrix}$$ (inside-mixed)

cf) Pauli matrix

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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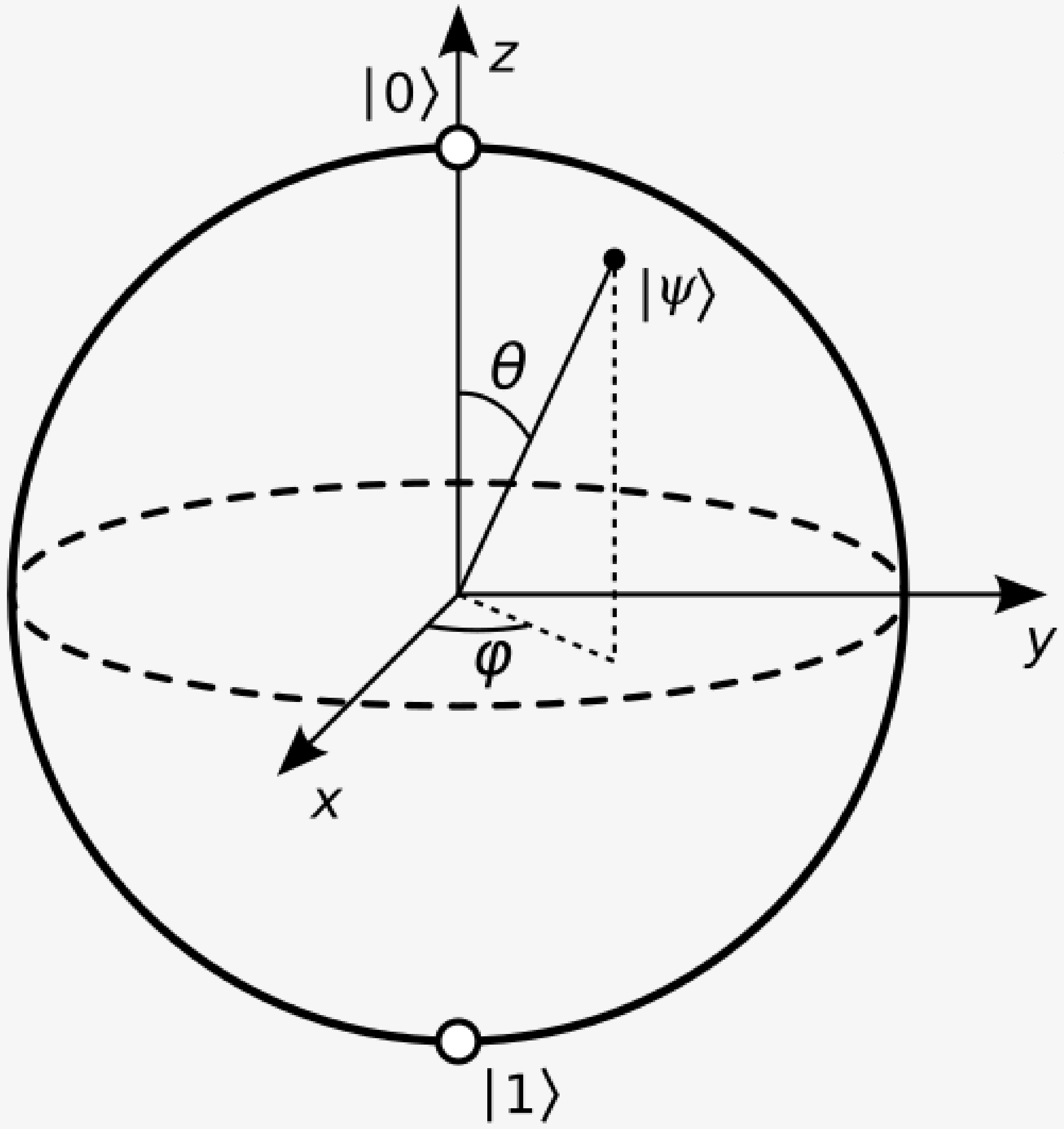
02. Bloch sphere

3) Bloch sphere (cont.)



rotation

$$R_x(\theta) = e^{-i\frac{\theta}{2}\sigma_x} = \begin{bmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

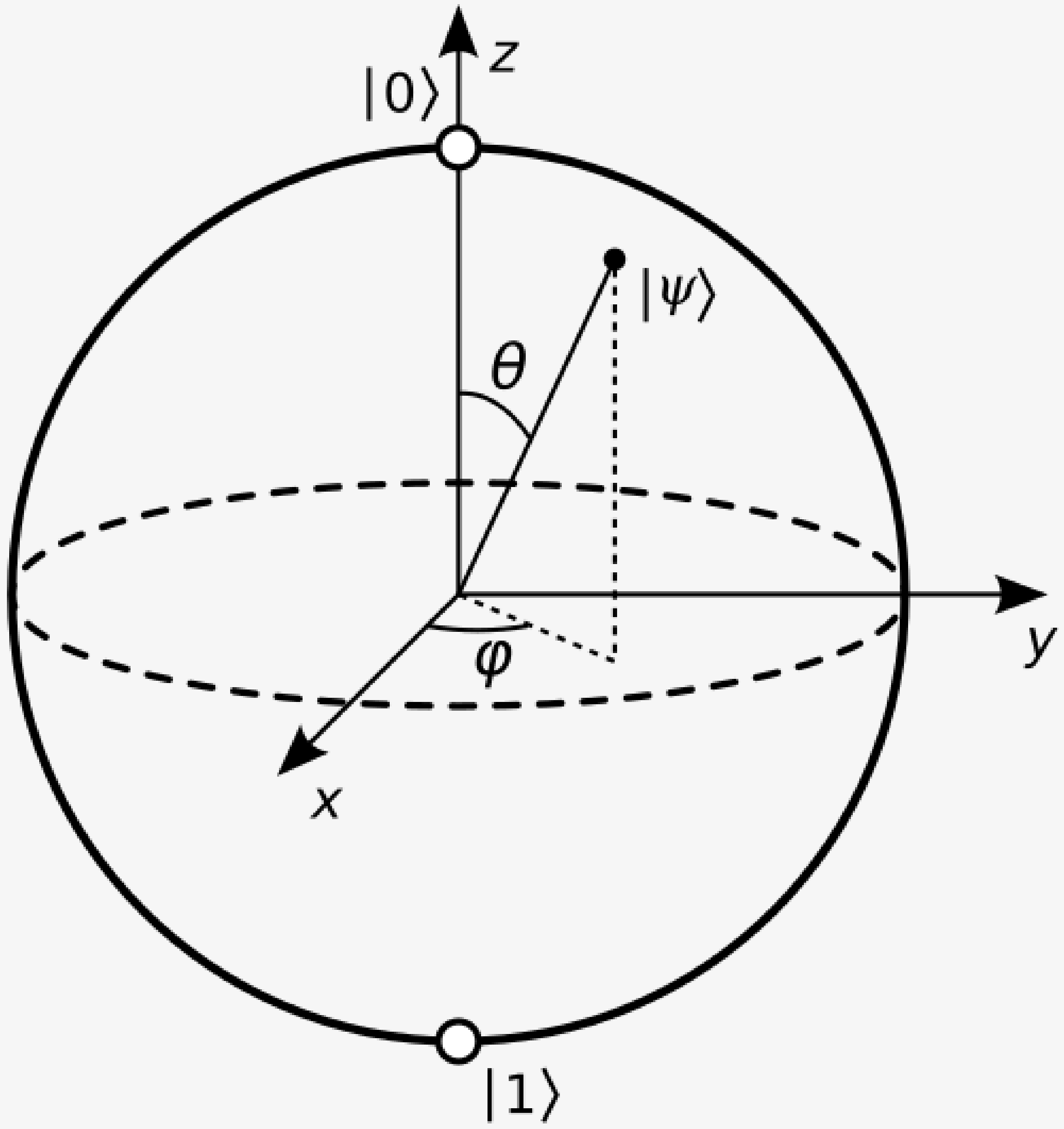
$$R_y(\theta) = e^{-i\frac{\theta}{2}\sigma_y} = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

$$R_z(\theta) = e^{-i\frac{\theta}{2}\sigma_z} = \begin{bmatrix} e^{-i\theta 2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

$$R_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}} = \cos(\theta/2) - i\sin(\theta/2)\hat{n}\cdot\vec{\sigma}$$

02. Bloch sphere

cf) Taylor expansion



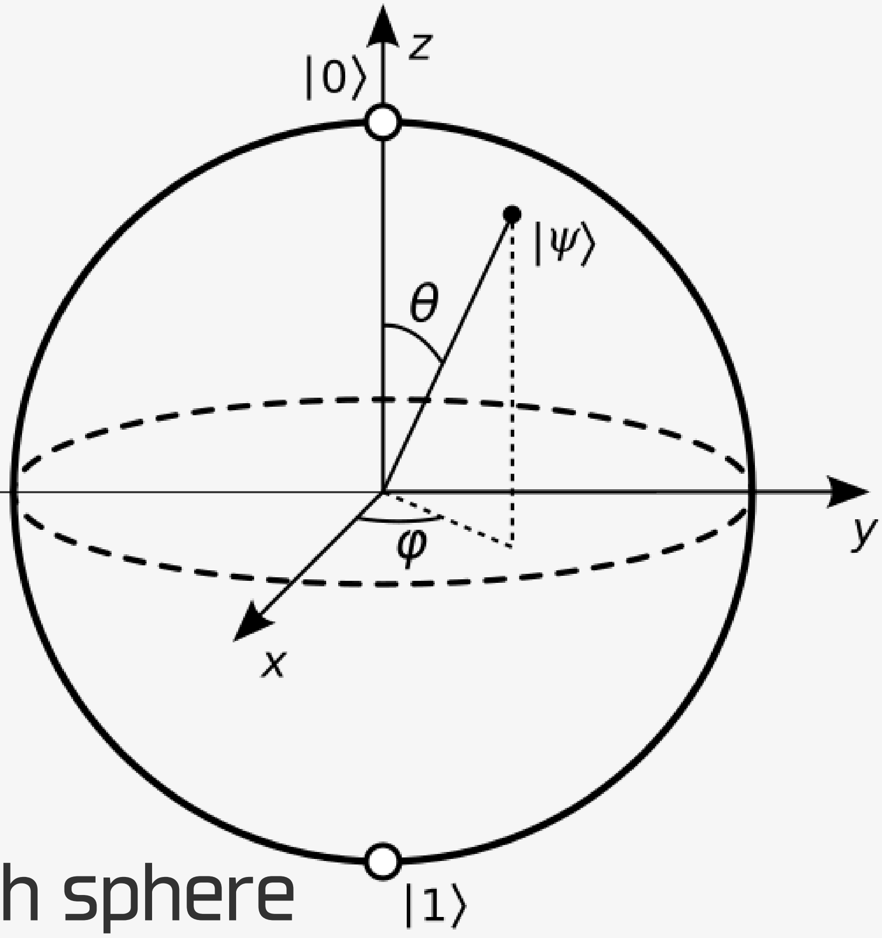
ex) $$R_x(\theta) = e^{-i\frac{\theta}{2}\sigma_x} = \begin{bmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_x^2 = I \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-i\frac{\theta}{2}\sigma_x} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-i\frac{1}{2}\theta\right)^n \sigma_x^n$$

$$e^{-i\frac{\theta}{2}\sigma_x} = \begin{bmatrix} 1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 + \cdots & -i\left(\frac{\theta}{2}\right) + i\frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \cdots \\ -i\left(\frac{\theta}{2}\right) + i\frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \cdots & 1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 + \cdots \end{bmatrix}$$

03. Qubits & Gates



1) Clbits vs. Qubits

- Binary Digits
- Represent true/false, on/off, normal/abnormal, etc.
- (exclusive state).

$$\Sigma = \{0, 1\}^n$$

- Quantum state on Bloch sphere
- Complex

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

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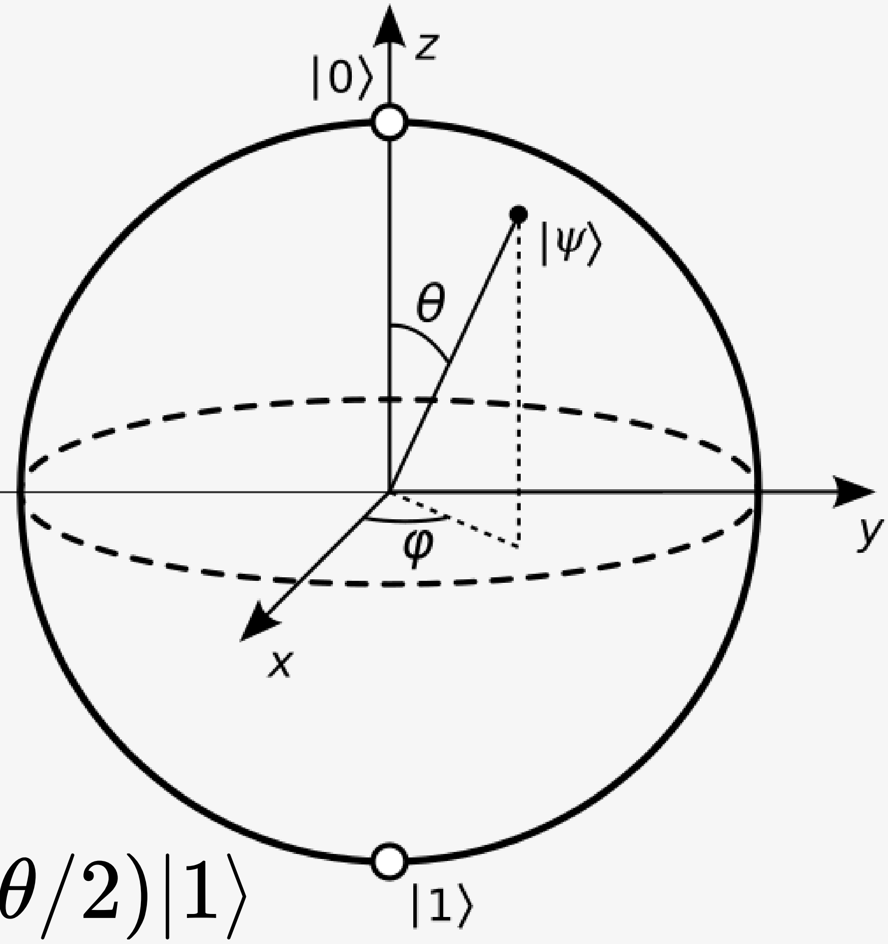
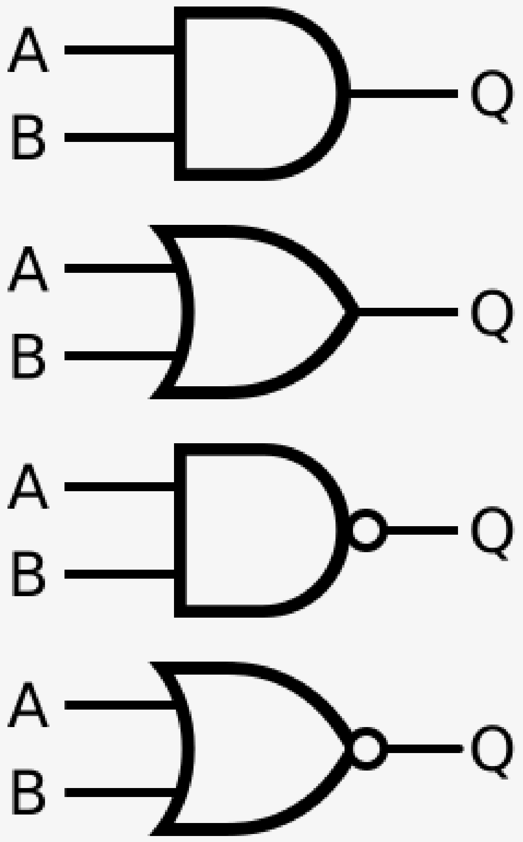
03. Qubits & Gates



2) Logic gate vs. Quantum Logic gate

$$\Sigma = \{0, 1\}^n$$

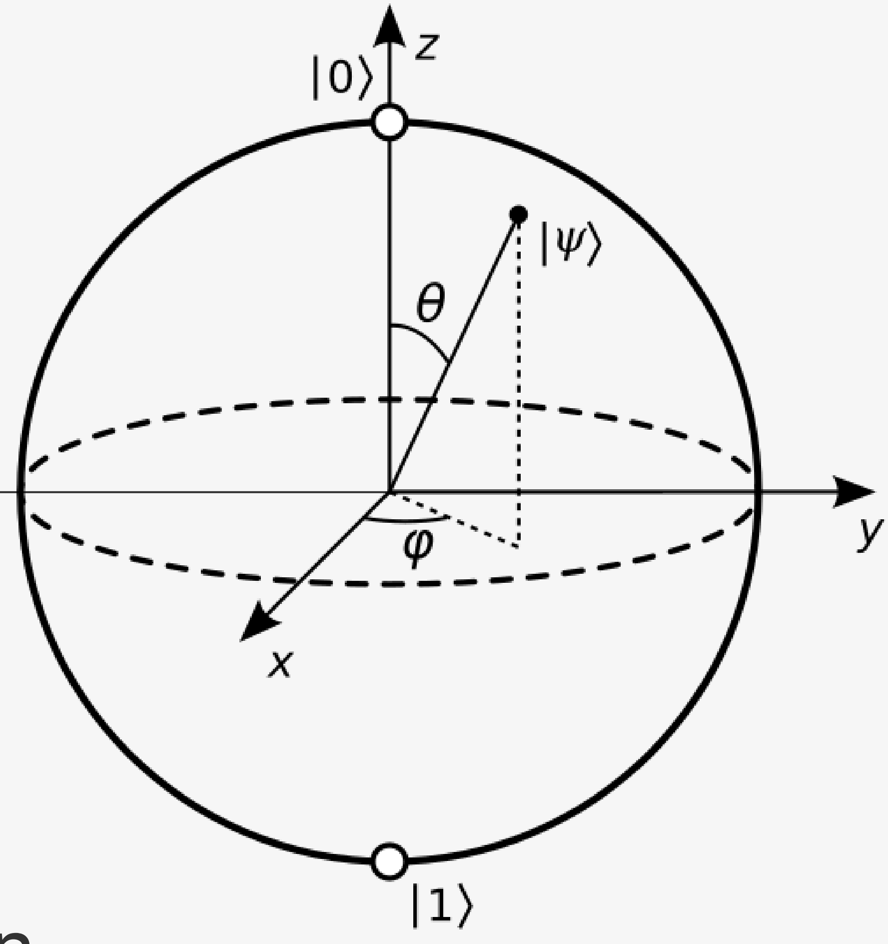
- Logic gates(and, or, nand, nor, ...)



$$|\psi\rangle = \cos{(\theta/2)}|0\rangle + e^{i\phi}\sin{(\theta/2)}|1\rangle$$

- Universal gate set
- Clifford set({CNOT, H, S}) + T-gate
- Rotation + CNOT
- Toffoli + H

03. Qubits & Gates



3) Quantum Logic gate (1-qubit)

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

- Hadamard

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

- Phase

$$P(\phi) = \begin{bmatrix}1 & 0\\ 0 & e^{i\phi}\end{bmatrix} = e^{-i\frac{\phi}{2}}\begin{bmatrix}1 & 0\\ 0 & e^{i\phi}\end{bmatrix} = -iR_z(\phi)$$

$$Z = \begin{bmatrix}1 & 0\\ 0 & e^{i\pi}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = P(\pi)$$

$$S = \begin{bmatrix}1 & 0\\ 0 & e^{i\frac{\pi}{2}}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix} = P\left(\frac{\pi}{2}\right)$$

$$T = \begin{bmatrix}1 & 0\\ 0 & e^{i\frac{\pi}{4}}\end{bmatrix} = P\left(\frac{\pi}{4}\right)$$

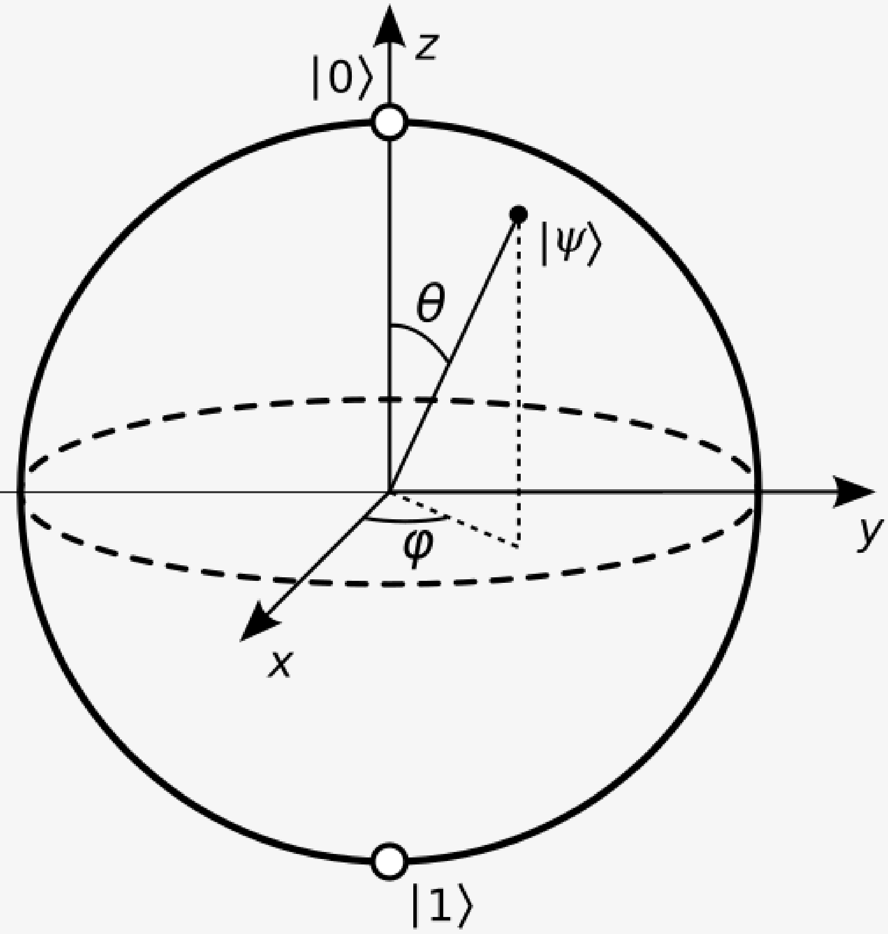
- rotation

$$R_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}}$$
$$= \cos(\theta/2) - i\sin(\theta/2)\hat{n}\cdot\vec{\sigma}$$

$$X = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = iR_x(\pi)$$

$$Y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = iR_y(\pi)$$

03. Qubits & Gates

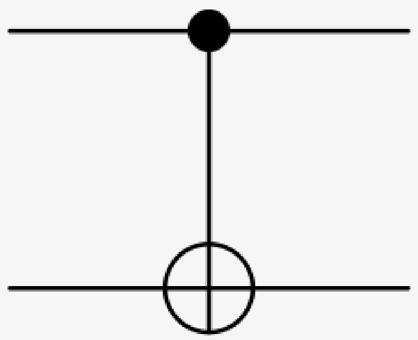


3) Quantum Logic gate (2-qubit)

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

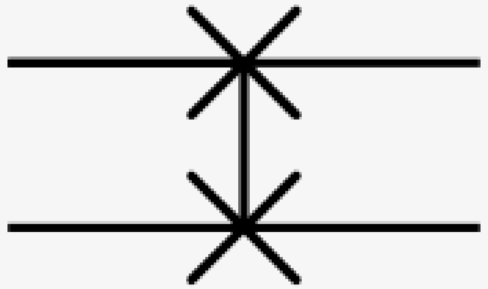
- CNOT(CX)

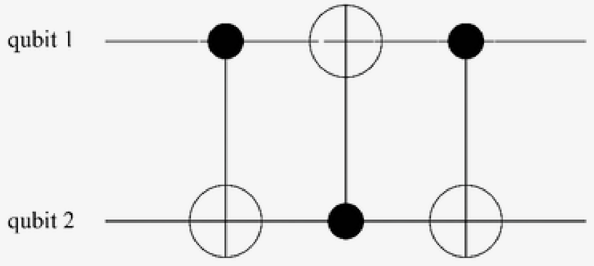
$$CNOT|a\rangle|b\rangle = |a\rangle|a \oplus b\rangle$$
$$CNOT = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

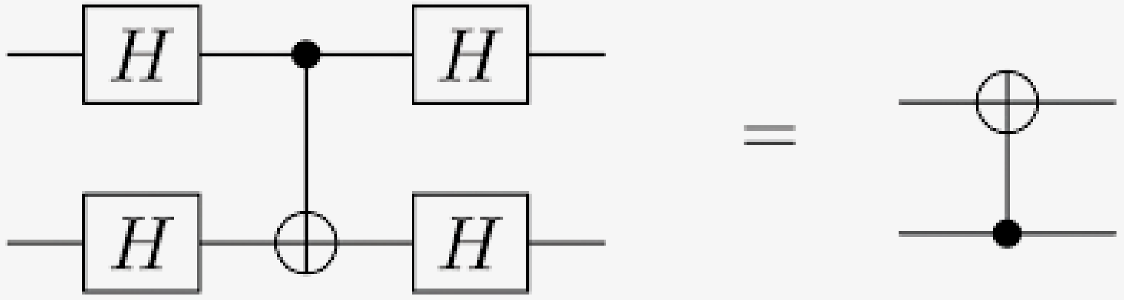


- SWAP

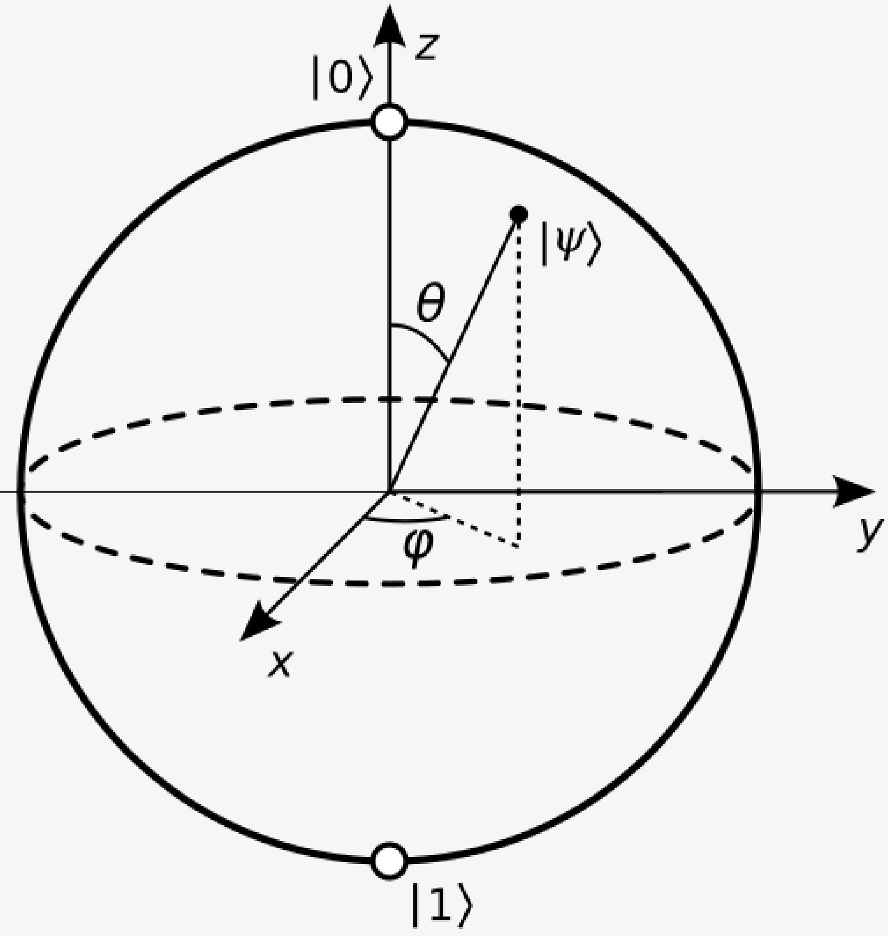
$$SWAP|a\rangle|b\rangle = |b\rangle|a\rangle$$
$$SWAP = |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



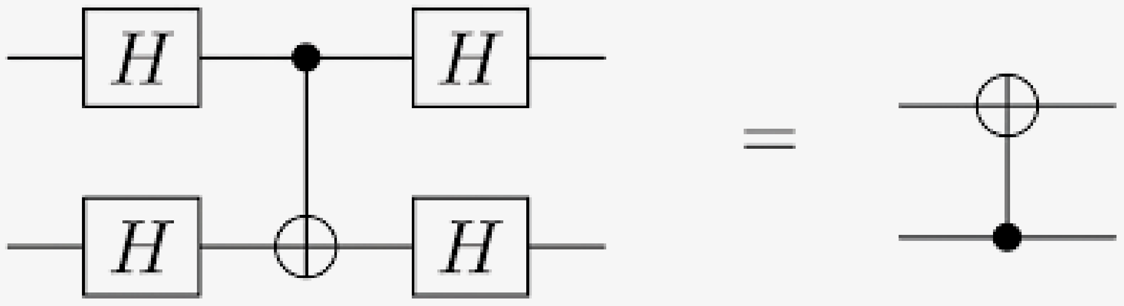




03. Qubits & Gates



cf) Sylvester's construction



$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

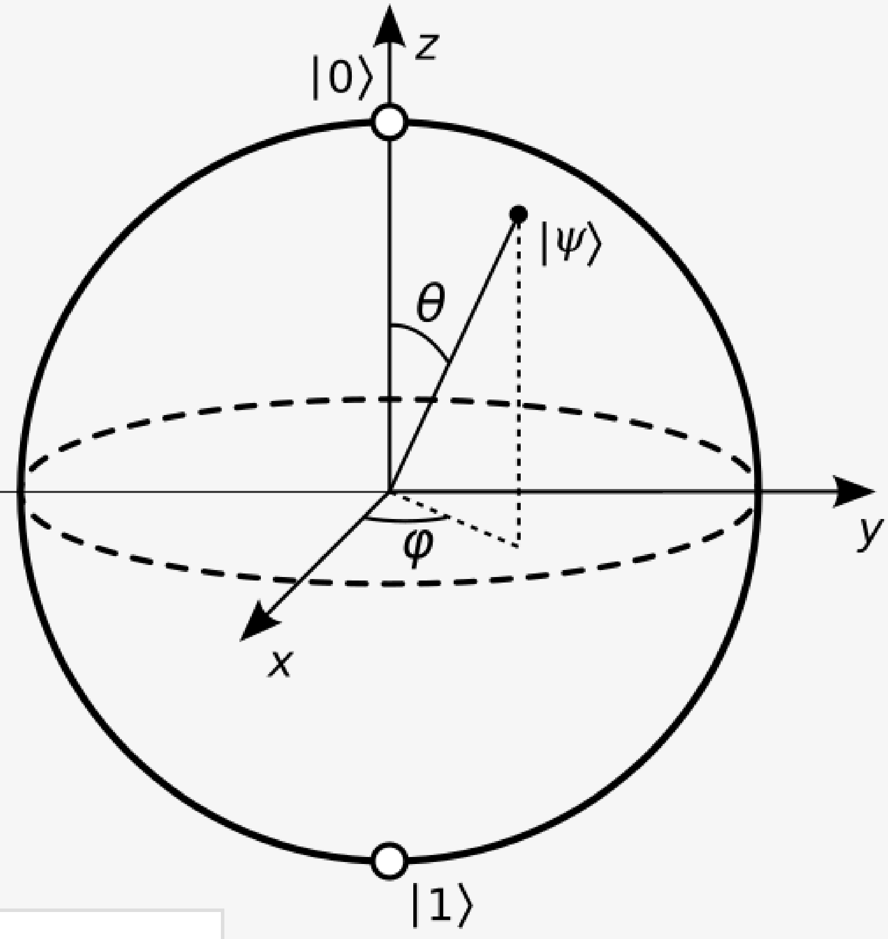
$$CNOT = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

$$= \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$$H_{2^k} = H_2 \otimes H_{2^{k-1}}$$

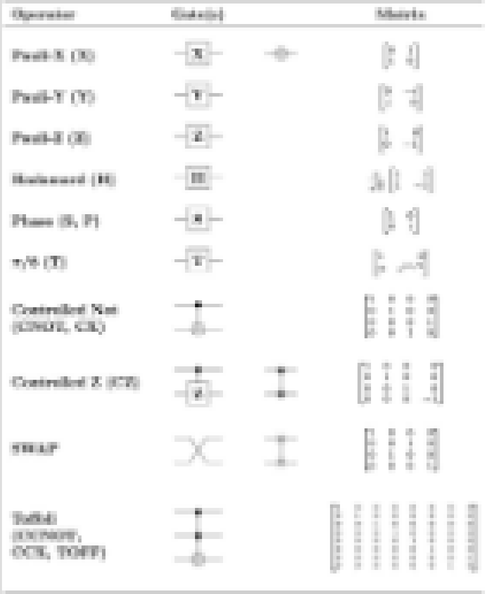
$$H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \longrightarrow H_4 = \frac{1}{\sqrt{2}}\begin{bmatrix}H_2 & H_2\\ H_2 & -H_2\end{bmatrix} = H_2 \otimes H_2$$

03. Qubits & Gates



4) Summary

Operator	Gate(s)		Matrix
Pauli-X (X)	X		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
Pauli-Y (Y)	Y		$\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$
Pauli-Z (Z)	Z		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Hadamard (H)	H		$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$
Phase (S, P)	S		$\begin{bmatrix}1 & 0\\0 & i\end{bmatrix}$
$\pi/8$ (T)	T		$\begin{bmatrix}1 & 0\\0 & e^{i\pi/4}\end{bmatrix}$
Controlled Not (CNOT, CX)			$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$
Controlled Z (CZ)	Z		$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}$
SWAP			$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$
Toffoli (CCNOT, CCX, TOFF)			$\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\end{bmatrix}$



Quantum logic gate

In quantum computing and specifically the quantum circuit model of computation, a quantum logic gate is a basic quantum…

W Wikipedia

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03. Qubits & Gates

cf) ???

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03. Qubits & Gates

cf) TOMORROW X TOGETHER (TXT)

투모로우바이투게더

투모로우바이투게더(영어: TOMORROW X TOGETHER · TXT)는 2019년 3월 4일에 데뷔한 대한민국 빅히트 뮤직 소속의 5인조 다국적 보이 그룹이다.

W Wikipedia

$$\simeq \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$$

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04. Superposition & Entanglement

1) Superposition

- linear, homogeneous PDE / eigenvalues & eigenvectors

$$\Psi = \sum_i c_i \psi_i \qquad \hat{A}\psi_i = \lambda_i \psi_i$$

- Quantum information processing

$$|\Psi\rangle = c_0|0\rangle + c_1|1\rangle \quad (\text{where } |c_0|^2 + |c_1|^2 = 1)$$

04. Superposition & Entanglement

1) Superposition

- Quantum information processing

$$|\Psi\rangle = c_0|0\rangle + c_1|1\rangle \quad (\text{where } |c_0|^2 + |c_1|^2 = 1)$$

ex) Hadamard for superposition

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

04. Superposition & Entanglement

2) Entanglement

- Pure state – separable tensor product
  $|\psi\rangle_a \otimes |\psi\rangle_b$

- Mixed state – no separable(entangled)

ex) Bell state

$$|\Phi\rangle_+ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi\rangle_- = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi\rangle_+ = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Psi\rangle_- = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

ex) GHZ(Greenberger–Horne–Zeilinger) state

$$GHZ = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

ex) W state

$$W = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

04. Superposition & Entanglement

2) Entanglement

- Mixed state - no separable(entangled)

ex) Bell state

$$|\Phi\rangle_+ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi\rangle_- = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi\rangle_+ = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Psi\rangle_- = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$|\psi\rangle_1 = a|0\rangle + b|1\rangle$$

$$|\psi\rangle_2 = c|0\rangle + d|1\rangle$$

$$\text{where } (|a|^2 + |b|^2 = 1, |c|^2 + |d|^2 = 1)$$

$$\begin{aligned} |\psi\rangle_1|\psi\rangle_2 &= (a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle) \\ &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle \end{aligned}$$

- impossible to make Bell states

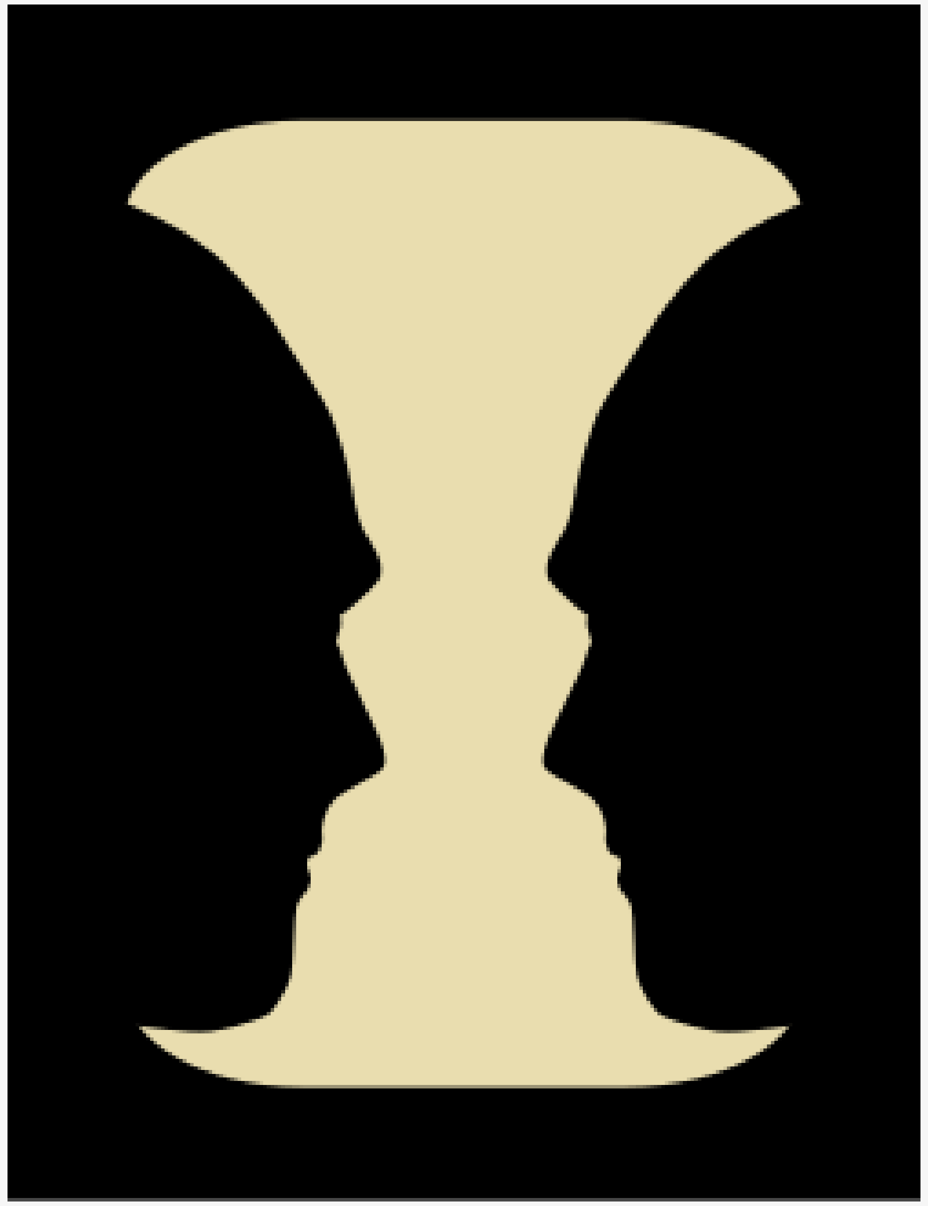
04. Superposition & Entanglement

2) Entanglement

It is possible to be pure state and mixed state concurrently, depending on which system you focus on.

ex)

$$|\Psi\rangle_{total} = |\Phi\rangle_{+}|+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$



04. Superposition & Entanglement

2) Entanglement

- Density matrix

By spectral theorem,

$$\rho = \sum_i w_i |\psi_i\rangle\langle\psi_i|$$

- pure state

$$\mathrm{Tr}(\rho^2) = 1$$

- mixed state

$$\mathrm{Tr}(\rho^2) < 1$$

ex) Bell state

$$\rho_{total} = |\Phi_+\rangle\langle\Phi_+| = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Tr}(\rho_{total}^2) = 1$$

- partial trace

$$\rho_{partial} = \frac{1}{2}I$$

$$\mathrm{Tr}(\rho_{partial}^2) = 1/2 < 1$$

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \rightarrow \begin{bmatrix} a+f & c+h \\ i+n & k+p \end{bmatrix}$$

Reference

- https://en.wikipedia.org/wiki/Unitary_matrix
- https://en.wikipedia.org/wiki/Hermitian_matrix
- https://en.wikipedia.org/wiki/Bloch_sphere
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- https://en.wikipedia.org/wiki/Quantum_entanglement
- https://en.wikipedia.org/wiki/Partial_trace
- https://en.wikipedia.org/wiki/W_state
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